AN ALTERNATIVE APPROACH FOR ENTERING VECTOR IN SIMPLEX METHOD

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(Received : April, 1980)

SUMMARY

In this paper an alternative approach for entering the variable in basis in simplex method is suggested. By considering an example, it is shown that the number of iterations may reduce.

INTRODUCTION

A Linear Programming Problem can be defined as

$$\text{Maximize} \sum_{j=1}^{n} c_j x_j \qquad \ldots(1)$$

$$\text{Subject to} \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i$$

$$i=1, 2 \ldots m$$

$$x_j \geqslant 0 \qquad j=1, 2 \ldots n \qquad \ldots(2)$$

In a linear programming problem any solution which satisfies the constraints (2) is called a feasible solution. A solution in whicn $n-m$ variables are set equal to zero is called a basic feasible solution. One of the basic feasible solutions may be optimal. So far simplex method is a wellknown method for solving such linear programming problems.

Following Rao [1], the entering vector is introduced in the basis according to the cost coefficients of objective function. The variable corresponding to maximum positive cost coefficient in the objective function is entered. The variable corresponding to the minimum positive ratio of constant on the right hand side to the corresponding cofficients of entering vector is droppcd.

The following are the transformations used for transforming the variables and cost coefficients in various iterations.

$$a_{ij}^{*}=a_{ij}-\frac{a_{ik}\,a_{rj}}{a_{rk}} \quad \text{for } i\neq r$$

$$j=1, 2 \ldots n$$

$$=\frac{a_{rj}}{a_{rk}} \quad \text{for } i=r$$

$$j=1, 2 \ldots n,$$

where a_{rk} is the pivotal element. Likewise new constant on the right hand side for equation i is

$$b_{i}^{*}=b_{i}-\frac{a_{ik}\,b_{r}}{a_{rk}} \quad \text{for } i\neq r$$

$$=\frac{b_{r}}{a_{rk}} \quad \text{for } i=r.$$

The new cost coefficients will be

$$c_{j}^{*}=c_{j}-\frac{a_{rj}\,c_{k}}{a_{rk}} \quad \text{for } j\neq k$$

and $c_{k}^{*}=0$.

ALTERNATIVE APPROACH :

Here we obtain maximum levels for each variable putting other variables equal to zero, for each constraint. We will be getting mn n-vectors as follows :

$$\left(0, 0 \ldots 0, x_{j}=\frac{b_{i}}{a_{ij}}, 0 \ldots 0\right)$$

$$i=1, 2 \ldots m$$

$$j=1, 2 \ldots n$$

Let us define

$$\hat{x}_{j}=\min_{1\leqslant i\leqslant m}\left(\frac{b_{i}}{a_{ij}}\right) \quad \text{for } a_{ij}>0$$

$$=0 \qquad \text{for } a_{ij}<0 \ \forall\ i$$

Then $(\hat{x}_{1}, 0 \ldots 0)$, $(0, \hat{x}_{2}\ 0 \ldots 0)$. $(0, 0, 0, \ldots 0, \hat{x}_{n})$, are n feasible solutions.

We define

$$y_{j}=c_{j}\,\hat{x}_{j},$$

So, y_j gives the effect of j^{th} variable on the objective function. Introduce the variable corresponding to the maximum value of y_j, say y_k, that is

$$y_k = \max_{1 \leqslant j \leqslant m} y_j,$$

then x_k is the entering variable and the outgoing variable is decided as usual. In the next iterations transformed effects are found using the formula

$$y_j^* = y_j - \frac{y_k\, a_{rj}}{a_{rk}} \quad j \neq k$$

and $\quad y_k^* = 0.$

The transformation of other entries are done as usual. When all the effects are non-positive the optimality is attained.

Identifying an optimal points

Theorem : A basic feasible solution is an optimal solution with a maximum objective function if all the y_j's are non-positive.

Proof : Suppose after some iteration we arive at the following stage.

$$\begin{array}{l} 1x_1 + 0x_2 + \ldots + 0x_m + a_{1,m+1}^{*} x_{m+1} + \ldots + a_{1n}^{*} x_n = b_1^{*} \\ 0x_1 + 1x_2 + \ldots + 0x_m + a_{2,m+1}^{*} x_{m+1} + \ldots + a_{2n}^{*} x_n = b_2^{*} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0x_1 + 0x_2 + \ldots + 1x_m + a_{m,m+1}^{*} x_{m+1} + \ldots + a_{mn}^{*} x_n = b_m^{*} \end{array} \quad \ldots(1)$$

$$y_j^* : 0 \quad 0 \quad \ldots \quad y_{m+1}^* \quad \ldots \quad y_n^*$$

where, $y_{m+1}^* \ldots y_n^*$ are non-positive.

Now, the basic solution which can be readily deduced from (1) is

$$x_i = b_i^* \qquad i = 1, 2 \ldots m$$

and $\quad x_i = 0 \qquad i = m+1, \ldots n$

And the value of the objective function is

$$z = \sum_{i=1}^{m} c_i x_i$$

Since here the variables $x_{m+1}, x_{m+2}, \dots x_n$ are zero and are constrained to be nonnegative, the only way one of them can change is to become positive. But increasing any variable x_i, $i=m+1, \dots n$ cannot increase the value of the objective function because the level of effects on the objective function y_j^*, $j=m+1, \dots n$ are nonpositive. Therefore the present solution must be optimal because no change in nonbasic variables can cause an increase in the value of the objective function.

EXAMPLE

It will be shown, by taking an example, that the number of iterations is reduced by using the proposed method.

$$\text{Max. } 4x_1+4x_2+1x_3+11x_4$$

Subject to,

$$x_1+x_2+x_3+x_4 \leqslant 15$$

$$7x_1+5x_2+3x_3+2x_4 \leqslant 120$$

$$3x_1+5x_2+10x_3+15x_4 \leqslant 100$$

$$x_i \geqslant 0, \; i=1, 2, 3, 4$$

After introducing the necessary slack variables the above linear programming problem can be written as

$$\text{Max. } 4x_1+4x_2+9x_3+11x_4+0x_5+0x_6+0x_7$$

Subject to,

$$x_1+x_2+x_3+x_4+x_5 = 15$$

$$7x_1+5x_2+3x_3+2x_4 + x_6 = 120$$

$$3x_1+5x_2+10x_3+15x_4 + x_7 = 100$$

$$x_i \geqslant 0, \; i=1, 2, \dots 7$$

By using the usual method, it can be seen that four iterations are required to reach the optimal solution. The optimal solution is $x_1=\frac{50}{7}$, $x_2=0$, $x_3=\frac{55}{7}$, $x_4=0$, $x_5=0$, $x_6=\frac{325}{7}$ $x_7=0$ and the optimal value of the objective function is $\frac{695}{7}$. It will be shown that only three iterations are required to achieve the optimality.

Here, $\hat{x}_1=15$, $\hat{x}_2=15$, $\hat{x}_3=10$, $\hat{x}_4=\frac{20}{3}$

The following table shows the calculations involved.

Basic variables	x_1	x_2	x_3	x_4	x_5	x_6	x_7	*Solution*
x_5	1	1	1	1	1	0	0	15
x_6	7	5	3	2	0	1	0	120
x_7	3	5	$\boxed{10}$	15	0	0	1	100→
y_j	60	60	90 ↑	$\frac{200}{3}$	0	0	0	
x_5	$\boxed{\frac{7}{10}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	$-\frac{1}{10}$	5→
x_6	$\frac{61}{10}$	$\frac{7}{2}$	0	$-\frac{5}{2}$	0	1	$-\frac{3}{10}$	90
x_3	$\frac{3}{10}$	$\frac{1}{2}$	1	$\frac{3}{2}$	0	0	$\frac{1}{10}$	10
y_j	33 ↑	15	0	$-\frac{185}{3}$	0	0	-9	
x_1	1	$\frac{5}{7}$	0	$-\frac{5}{7}$	$\frac{10}{7}$	0	$-\frac{1}{7}$	$\frac{50}{7}$
x_6	0	$\frac{95}{14}$	0	$\frac{13}{7}$	$-\frac{61}{7}$	1	$\frac{4}{7}$	$\frac{325}{7}$
x_3	0	$\frac{1}{14}$	1	$\frac{12}{7}$	$\frac{3}{7}$	0	$\frac{1}{7}$	$\frac{55}{7}$
y_j	0	$-\frac{60}{7}$	0	$-\frac{800}{21}$	$-\frac{300}{7}$	0	$-\frac{30}{7}$	

Remark : The new approach gives the maximum increase at the first iteration. Therefore, in some problems, the method may reduce the number of iterations. Moreover, it is observed that in no problem number of iterations by the new approach is more than that in the usual method.

ACKNOWLEDGEMENTS :

The author is grateful to the referees for their valuable comments in improving the paper to its present form.

REFERENCE

[1] S.S. RAO (1978) : Optimization theory and applications, Wiley Eastern Limited.